THE KING'S SCHOOL

2008
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

Blank Page

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the remainder when the polynomial $P(x) = x^5 - 2x^3 + 11$ is divided by $x + 1$ **2**

(b) The interval AB has end points $A(-1, 8)$ and $B\left(\frac{10}{3}, \frac{28}{3}\right)$

Find the point $P(x, y)$ which divides AB externally in the ratio 3:2. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{1}{4}} \frac{12}{\sqrt{1 - 4x^2}}\, dx$ **3**

(d) Find in simplest form the derivative of $\dfrac{x}{1 + x^2} + \tan^{-1} x$ **3**

(e) By writing $2x$ as $(x + y) + (x - y)$, or otherwise, simplify $\dfrac{2x}{x^2 - y^2} - \dfrac{1}{x - y}$ **2**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Find the remainder when $3x^4 + 2x^2 + 2x - 1$ is divided by $x^2 + 1$ **2**

(ii) Hence find $\int \frac{3x^4 + 2x^2 + 2x - 1}{x^2 + 1}\, dx$ **1**

(b) Use the substitution $u = 1 - x$ to evaluate $\int_0^1 360x(1 - x)^4\, dx$ **3**

(c) A particle moves on the x axis with velocity v given by $v = (x + 1)^2$. Initially the particle is at the origin. Find the initial acceleration. **3**

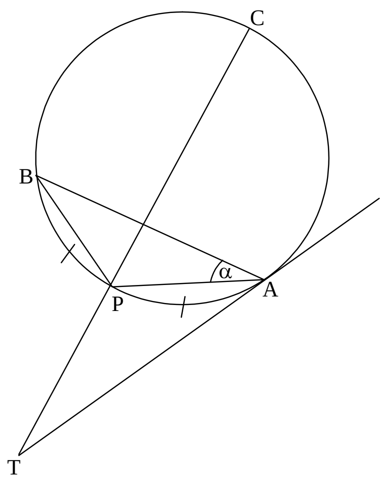
(d) Solve the inequality $\frac{3x - 1}{2x + 3} > 1$ **3**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $12\sin^{-1} x = \cos^{-1} x$, giving your answer correct to two decimal places. **3**

(b)



TA is a tangent at A to the circle. TPC is a secant to the circle and B is chosen on the circle so that arc PA = arc PB. Let $\angle PAB = \alpha$

(i) Explain why $\angle PBA = \alpha$ **1**

(ii) Prove that AP bisects $\angle TAB$ **2**

(c) A particle is moving in simple harmonic motion on the *x* axis according to the equation of motion $x = -12\cos nt$, where $t \geq 0$ is time and $n > 0$.
The period of the motion is *T*.

(i) Prove that the particle first reaches the position $x = 6$ when the time is $\frac{T}{3}$ **3**

(ii) Find the velocity of the particle when $t = \frac{T}{3}$ **2**

(iii) Write down the time that will elapse after $t = \frac{T}{3}$ for the particle to be next at rest. **1**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle is falling vertically so that at any time t s its velocity v m/s is given by $\frac{dv}{dt} = 10 - 0.2v$. Initially $v = 20$.

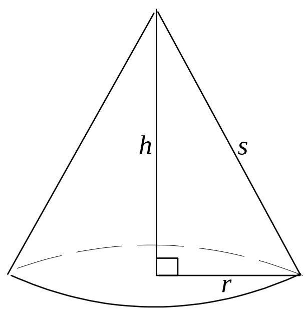
(i) Verify by differentiation that $v = 50 + Ae^{-0.2t}$ for some constant A. **2**

(ii) Find A. **1**

(iii) Find the time taken, correct to one decimal place, for the velocity to reach 40 m/s. **2**

(iv) Find the distance travelled during the first 5 seconds. **2**

(b)



The diagram shows a cone with base radius r, height h and slant height s.

A conical pile is being formed at a constant rate of 2m^3/min.

The pile at any time t is such that $h = \frac{4}{3}r$.

(i) Show that $s = \frac{5r}{3}$ **1**

(ii) Find the rate of increase of the curved surface area at the instant the radius is 5m.

$$[V = \frac{1}{3}\pi r^2 h, \text{ Curved Surface Area } = \pi rs]$$ **4**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the coefficient of x^9 in the binomial expansion of $\left(2x + \frac{3}{x^2}\right)^{30}$

[LEAVE YOUR ANSWER IN UNSIMPLIFIED FORM] **4**

(b) (i) Sketch $y = \sin x$ and $y = \cos x$ on the same axes for $0 \le x \le \frac{\pi}{2}$, clearly showing their point of intersection. **2**

(ii) The region enclosed between $y = \sin x$ and $y = \cos x$ and the vertical lines $x = 0$ and $x = \frac{\pi}{2}$ is revolved about the x axis.

Find the volume of the solid generated. **3**

(c) Prove by induction for integers $n \ge 1$ that

$4.1 + 8.3 + 12.3^2 + \ldots + 4n.3^{n-1} = (2n-1)3^n + 1$ **3**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

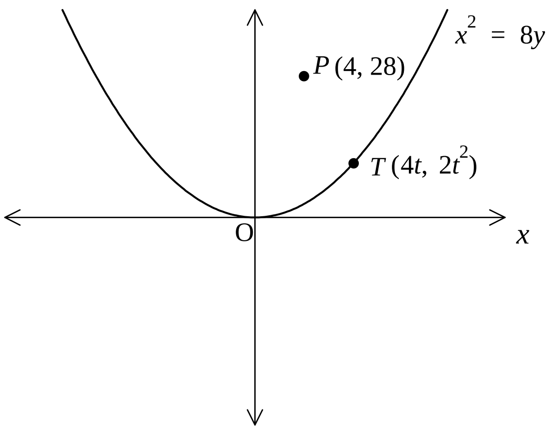
(a) Let $f(t) = t^3 - 12t - 2$.

Since $f(-1) = 9$ and $f(0) = -2$ there is a root of $f(t) = 0$ between $t = -1$ and $t = 0$.

[DO NOT EXPLAIN THE REASON]

Use Newton's Method once with a trial root of $t = -0.2$ to give a two decimal approximation to the root between $t = -1$ and t = 0 . **3**

(b)



The diagram shows the parabola $x^2 = 8y$ and the point $P(4, 28)$ interior to the parabola. Let $T(4t, 2t^2)$ be any point on the parabola.

(i) Prove that the equation of the normal at T is $x + ty = 4t + 2t^3$ **2**

(ii) The normal at T passes through $P(4, 28)$. Show that $t^3 - 12t - 2 = 0$ **1**

(iii) Deduce that if there are three normals which can pass through $P(4, 28)$ then the sum of the x coordinates of the points $T(4t, 2t^2)$ is zero. **2**

(iv) Find the sum of the y coordinates of the points $T(4t, 2t^2)$ in (iii). **2**

(v) Prove that there are three normals which pass through $P(4, 28)$. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(x) = e^{x} - e^{-x}$

(i) Show that the function is an odd function . **1**

(ii) Show that $f(x)$ increases as x increases for all values of x. **1**

(iii) Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram and include the line $y = x$. **2**

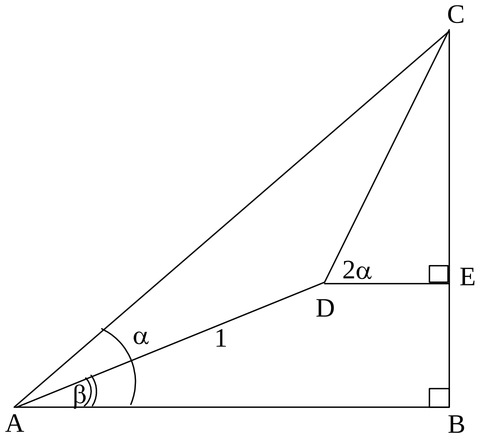
(iv) Show that $e^{\ln x} = x$ **1**

(v) It can be shown that $f^{-1}(x) = \ln\left(\dfrac{x + \sqrt{x^2 + 4}}{2}\right)$

[DO NOT SHOW THIS]

Show that $\displaystyle\int_0^2 f^{-1}(x)\, dx = 2\ln(1 + \sqrt{2}) + 2 - 2\sqrt{2}$ **3**

(b)



In the diagram
AD = 1, $\angle$BAC = α, $\angle$BAD = β and $\angle$EDC = 2α,
$\angle$CED = $\angle$CBA = 90°

(i) Find $\angle$ACD. **1**

(ii) Prove that BC = $\sin(2\alpha - \beta)$. **3**

End of Examination

Blank Page

Blank Page

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

THE KING'S SCHOOL

2008
Higher School Certificate
Trial Examination

Mathematics Extension 1

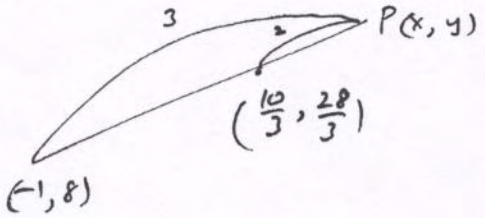
Question	Algebra and Number		Geometry		Functions		Trigonometry		Differential Calculus		Integral Calculus		Total
1	(e)	2			(a), (b)	4			(d)	3	(c)	3	12
2	(d)	3			(a)(i)	2			(c)	3	(a)(ii), (b)	4	12
3			(b)	3	(a)	3	(c)	6					12
4					(a)(ii)(iii)	3	(a)(i)	2	(b)	5	(a)(iv)	2	12
5	(a), (c)	7					(b)(i)	2			(b)(ii)	3	12
6					(a), (b)	12							12
7			(b)(i)	1	(a)	8	(b)(ii)	3					12
Total		12		4		32		13		11		12	84

TKS EXTENSION 1 SOLUTIONS TRIAL 2008

Question 1

(a) $R = P(-1) = -1 + 2 + 11 = 12$

(b)



$\therefore x = \dfrac{3 \cdot \frac{10}{3} - 2(-1)}{3-2} = 10 + 2 = 12$

$y = 28 - 16 = 12$

i.e. $P = (12, 12)$

(c) $I = \dfrac{12}{2}\left[\sin^{-1} 2x\right]_0^{\frac{1}{4}} = 6\left(\dfrac{\pi}{6} - 0\right) = \pi$

(d) $\dfrac{1 + x^2 - x(2x)}{(1+x^2)^2} + \dfrac{1}{1+x^2}$

$= \dfrac{1 - x^2}{(1+x^2)^2} + \dfrac{1+x^2}{(1+x^2)^2} = \dfrac{2}{(1+x^2)^2}$

(e) $\dfrac{x+y+x-y}{(x-y)(x+y)} - \dfrac{1}{x-y} = \dfrac{1}{x-y} + \dfrac{1}{x+y} - \dfrac{1}{x-y} = \dfrac{1}{x+y}$

or $\dfrac{2x}{(x-y)(x+y)} - \dfrac{x+y}{(x-y)(x+y)}$

$= \dfrac{x-y}{(x-y)(x+y)} = \dfrac{1}{x+y}$

Question 2

(a) (i)

$$\begin{array}{r} 3x^2 \quad\quad\quad\quad -1 \\ x^2+1 \overline{\smash{)}\, 3x^4 \quad +2x^2 \quad +2x \quad -1} \\ 3x^2 \quad\quad\quad -1 \\ \overline{-x^2} \quad\quad\quad \overline{0} \end{array}$$

$\therefore$ remainder $= 2x$

(ii) $I = \int 3x^2 - 1 + \frac{2x}{x^2+1}\, dx = x^3 - x + \ln(x^2+1)$

(b) $u = 1 - x$; $x = 0, u = 1$

$\frac{du}{dx} = -1$; $x = 1, u = 0$

$\therefore I = -\int_1^0 360\,(1-u)\,u^4\, du$

$= 360 \int_0^1 u^4 - u^5\, du = 360 \left[\frac{u^5}{5} - \frac{u^6}{6} \right]_0^1$

$= 360 \left(\frac{1}{5} - \frac{1}{6} \right) = 360 . \frac{1}{30} = 12$

(c) $\frac{1}{2} v^2 = \frac{1}{2} (x+1)^4$

$\therefore \ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = 2(x+1)^3$

$\therefore$ initial accel^n at $x = 0$ is 2

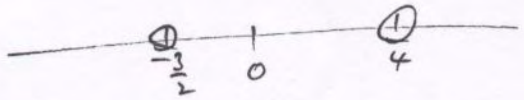
(d) (LOTS OF ALTERNATIVES) MOST COMMON ....

$2x + 3 \neq 0$ i.e. $x \neq -\frac{3}{2}$

Put $\frac{3x-1}{2x+3} = 1 \Rightarrow 3x - 1 = 2x + 3$

$x = 4$

$\Rightarrow$



Try $x = 0 \Rightarrow -\frac{1}{3} > 1$ ✗

$\therefore x < -\frac{3}{2}$ or $x > 4$

Question 3

(a) $\therefore\ 12\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}x$

$13\sin^{-1}x = \frac{\pi}{2} \Rightarrow \sin^{-1}x = \frac{\pi}{26}$

or $x = \sin\frac{\pi}{26} = 0{\cdot}12$, 2 d.p.

(b) (i) $\angle PBA = \angle PAB = \alpha$, angles at circumference standing on same size arc, $PA = PB$

(ii) $\angle TAP = \angle PBA = \alpha$, alternate segment thm

But $\angle PAB = \alpha$, data

$\therefore$ AP bisects $\angle TAB$

(c) (i) $T = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{T}$

$\therefore\ 6 = -12\cos\frac{2\pi}{T}t \Rightarrow \cos\frac{2\pi}{T}t = -\frac{1}{2}$

$\therefore\ \frac{2\pi}{T}t = \frac{2\pi}{3}$ for first t

$\Rightarrow t = \frac{T}{3}$

(ii) $v = -12(-n\sin nt) = \frac{24\pi}{T}\sin\frac{2\pi}{3}$

$= \frac{24\pi}{T}\cdot\frac{\sqrt{3}}{2} = \frac{12\sqrt{3}\,\pi}{T}$

(iii) since period $= T$ the time that will elapse

$= \frac{T}{2} - \frac{T}{3} = \frac{T}{6}$

Question 4

(a) (i) If $v = 50 + Ae^{-0.2t}$ then

$$\frac{dv}{dt} = -0.2Ae^{-0.2t} = -0.2(v-50)$$

$$= 10 - 0.2v$$

(ii) $t=0, v=20 \Rightarrow 20 = 50 + A, \quad A = -30$

(iii) $\therefore 40 = 50 - 30e^{-0.2t}$

$$e^{-0.2t} = \frac{1}{3}$$

or $e^{0.2t} = 3$

$\therefore 0.2t = \ln 3 \Rightarrow t = 5.5$ s, 1 d.p.

(iv) $$x = \int_0^5 50 - 30e^{-0.2t}\,dt$$

$$= \left[50t + 150e^{-0.2t}\right]_0^5$$

$$= 250 + 150e^{-1} - 150$$

$$= 100 + 150e^{-1} \text{ m}$$

[= 155 m, nearest metre]

(b) (i) $s^2 = h^2 + r^2 = \frac{16r^2}{9} + r^2 = \frac{25r^2}{9} \Rightarrow s = \frac{5r}{3}$

(ii) $A = \pi r s = \pi r \cdot \frac{5r}{3} = \frac{5}{3}\pi r^2 \quad \therefore \frac{dA}{dr} = \frac{10}{3}\pi r$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \cdot \frac{4}{3}r = \frac{4}{9}\pi r^3 \quad \therefore \frac{dV}{dr} = \frac{4}{3}\pi r^2$

$\therefore \frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dV}\cdot\frac{dV}{dt}$ where $\frac{dV}{dt} = 2$

$= \frac{10}{3}\pi r \cdot \frac{3}{4\pi r^2} \cdot 2 = \frac{5}{r} = 1 \text{ m}^2/\text{min}$ when $r = 5$

Question 5

(a) $$u_{k+1} = \binom{30}{k}(2x)^{30-k}\left(\frac{3}{x^2}\right)^k$$

$$= \binom{30}{k} 2^{30-k} 3^k \frac{x^{30-k}}{x^{2k}}$$

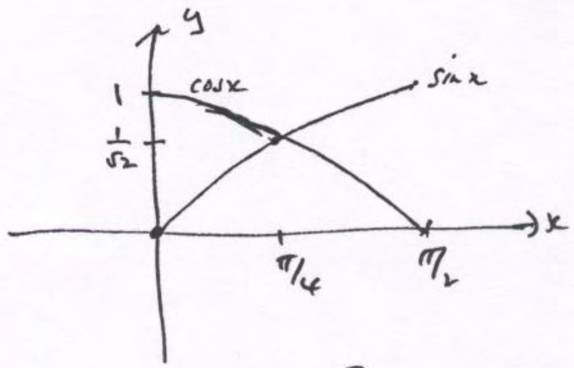
$$= \binom{30}{k} 2^{30-k} 3^k x^{30-3k}$$

For coefficient of x^9 we'd have $30-3k=9$

$\Rightarrow k=7$

$\therefore$ coefft of x^9 is $\binom{30}{7} 2^{23} 3^7$

(b) (i) $\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$ for $0 < x < \frac{\pi}{2}$



(ii) From sketch, $V = 2\pi \int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \, dx$

$$= 2\pi \int_0^{\pi/4} \cos 2x \, dx$$

$$= 2\pi \, \tfrac{1}{2} \left[\sin 2x\right]_0^{\pi/4}$$

$$= \pi(1-0) = \pi$$

(C) For $n=1$, $LS = 4 \times 1 = 4$

$RS = 1 \times 3 + 1 = 4$

$\therefore$ Assume $4.1 + 8.3 + \dots + 4n.3^{n-1} = (2n-1)3^n + 1$ for any integer $n \geq 1$

Then $4.1 + 8.3 + \dots + 4n.3^{n-1} + 4(n+1)3^n$

$= (2n-1)3^n + 1 + (4n+4)3^n$, using the assumption

$= 3^n(2n-1 + 4n+4) + 1$

$= 3^n(6n+3) + 1$

$= 3^{n+1}(2n+1) + 1 = RS$ for $n+1$

$\therefore$ if correct for n, it's correct for $n+1$

But it is correct for $n=1$

$\therefore$ by induction, $4.1 + \dots + 4n.3^{n-1} = (2n-1)3^n + 1$ for $n \geq 1$

Question 6

(a) $f'(t) = 3t^2 - 12$

$$\therefore t_1 = -0\cdot 2 - \frac{(-0\cdot 2)^3 - 12(-0\cdot 2) - 2}{3(-0\cdot 2)^2 - 12} = -0\cdot 17, \text{ 2 d.p.}$$

(b) (i) $y = \frac{x^2}{8}$ $\therefore \frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4} = t$ at $T(4t, 2t^2)$

$\therefore$ normal is $y - 2t^2 = -\frac{1}{t}(x - 4t)$

or $ty - 2t^3 = -x + 4t$

ie $x + ty = 4t + 2t^3$

(ii) Since $P(4, 28)$ is on the normal

$4 + 28t = 4t + 2t^3$

ie $t^3 - 12t - 2 = 0$

(iii) Sum of x coordinates $= 4(t_1 + t_2 + t_3)$, say,

where $t_1 + t_2 + t_3 = 0$, sum of roots of equation in (ii)

$\Rightarrow$ sum of x coordinates $= 0$

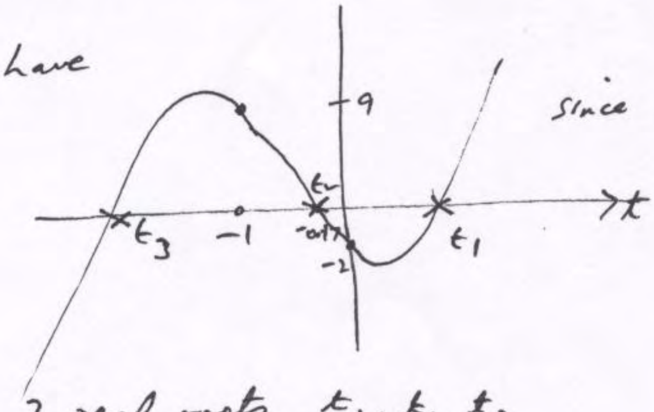
(iv) Sum $= 2(t_1^2 + t_2^2 + t_3^2)$

$= 2\left((t_1 + t_2 + t_3)^2 - 2(t_1t_2 + t_2t_3 + t_3t_1)\right)$

$= 2(0^2 - 2(-12)) = 48$

(v) Need to show $t^3 - 12t - 2 = 0$ has 3 real roots

From (a) we have



since for large $|t|$, $f(t) \sim t^3$

$\Rightarrow$ 3 real roots t_1, t_2, t_3

i.e. 3 normals.

Question 7

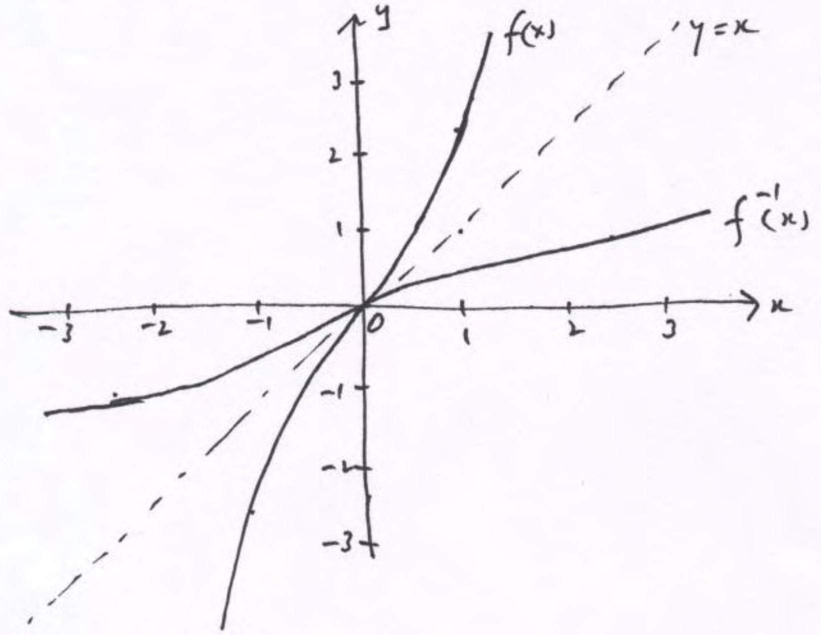
(a) (i) $f(-x) = e^{-x} - e^{x} = -(e^{x} - e^{-x}) = -f(x)$

$\therefore f(x)$ is an odd function

(ii) $f'(x) = e^{x} + e^{-x} > 0 \quad \forall x$

$\therefore f(x)$ is an increasing function $\forall x$

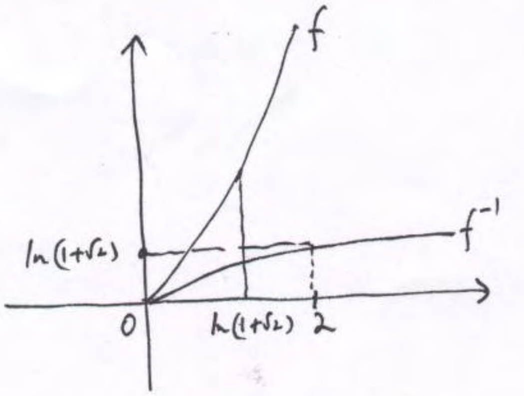
(iii) $f(0) = 0$, $f(1) = e - e^{-1} \simeq 2.35$, $f'(0) = 2$



(iv) Suppose $e^{\ln x} = N$

Then $\ln x = \log_e N \Rightarrow N = x$ i.e. result

(v) $f^{-1}(2) = \ln\left(\frac{2+\sqrt{8}}{2}\right) = \ln(1+\sqrt{2})$



$\therefore \int_0^2 f^{-1}(x)\,dx = 2\ln(1+\sqrt{2}) - \int_0^{\ln(1+\sqrt{2})} e^{x} - e^{-x}\,dx$

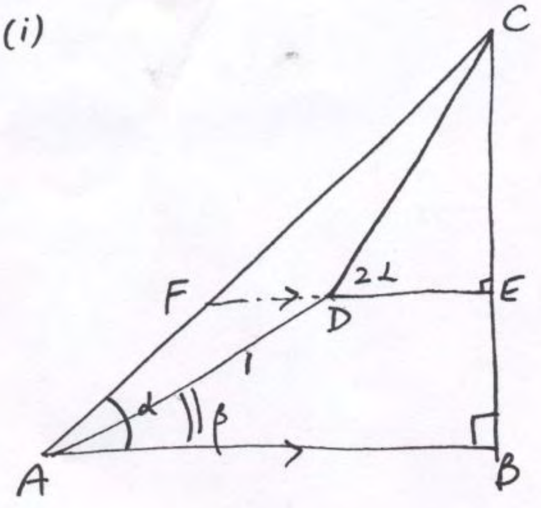
$= 2\ln(1+\sqrt{2}) - \left[e^{x} + e^{-x}\right]_0^{\ln(1+\sqrt{2})}$

$= 2\ln(1+\sqrt{2}) - \left(1+\sqrt{2} + \frac{1}{1+\sqrt{2}} - (1+1)\right)$

$= 2\ln(1+\sqrt{2}) - (1+\sqrt{2} + \sqrt{2} - 1 - 2)$

$= 2\ln(1+\sqrt{2}) + 2 - 2\sqrt{2}$

(b) (i)



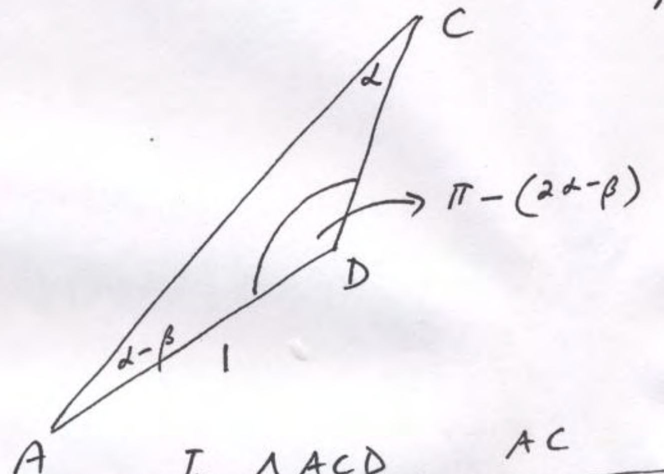
Produce ED to meet AC at F

Then $\angle CFE = \alpha$ $(= \angle CAB)$

$\therefore \angle ACD = \alpha$, ext $\angle$ thm in ΔFDC

[No reasons needed anywhere]
[MANY ALTERNATIVES]

(ii) In ΔCAB, $\sin\alpha = \dfrac{BC}{AC}$ i.e. $BC = AC\sin\alpha$



In ΔACD, $\dfrac{AC}{\sin(\pi-(2\alpha-\beta))} = \dfrac{1}{\sin\alpha}$

$$\Rightarrow AC = \frac{\sin(2\alpha-\beta)}{\sin\alpha}$$

$$\therefore BC = \frac{\sin(2\alpha-\beta)}{\sin\alpha}\cdot\sin\alpha = \sin(2\alpha-\beta)$$